Duality in $N = 2$ Super-Liouville Theory

Changrim Ahn[1†], Chanju Kim[2†], Chaiho Rim[3‡], and M. Stanishkov[4†]

[†] *Department of Physics, Ewha Womans University*
Seoul 120-750, Korea

[‡] *Department of Physics, Chonbuk National University*
Chonju 561-756, Korea

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Abstract

In this paper we consider a strong-weak coupling duality of the $N = 2$ super-Liouville field theory (SLFT). Without the self-duality found in other Liouville theories, the $N = 2$ SLFT, we claim, is associated with a 'dual' action by a transformation $b \to 1/b$ where b is the coupling constant. To justify our conjecture, we compute the reflection amplitudes (or two-point functions) of the (NS) and the (R) operators of the $N = 2$ SLFT based on the conjectured dual action and show that the results are consistent with known results.

1 Introduction

Two-dimensional Liouville field theory (LFT) appears naturally in the context of the 2D quantum gravity and string theories [1, 2]. This theory has been extended to the supersymmetric Liouville theories to accomodate world-sheet supersymmetries appearing in the string theories. These Liouville-type theories are also interesting for quantum field theoretical properties as well. They possess both conformal symmetry and the strong-weak coupling duality. The strong-weak coupling duality has been attracting much attention

[1]ahn@dante.ewha.ac.kr
[2]cjkim@phya.snu.ac.kr
[3]rim@mail.chonbuk.ac.kr
[4]stanishkov@dante.ewha.ac.kr; On leave of absence from Institute of Nuclear Research and Nuclear Energy, Sofia, Bulgaria

recently to understand non-perturbative aspects of the various quantum field theories rigorously. For example, the seminal work of Seiberg and Witten on the supersymmetric Yang-Mills theories in (3+1)-dimensions is based on the intuitive observation that the gauge theories in the strong coupling limit are described by a weak coupling region of some effective action [3]. More recently, similar duality arises in the string theory context to understand strings and branes nonperturbatively. This duality also arises in statistical systems such as the 2D Ising model.

For the LFT and the $N = 1$ SLFT, the strong-weak coupling duality appears as a quantum symmetry closely connected to (super-)conformal symmetries. It has been observed that the background charge is renormalized to $Q = b + 1/b$ by quantum corrections and the theories preserve the quantum conformal symmetries [2, 4]. This means the two LFTs are invariant under $b \to 1/b$, i.e. self-dual. These two symmetries are essential to determine exact correlation functions for the LFT [5] and the $N = 1$ SLFT [6]. Now let us consider the duality symmetry of the $N = 2$ SLFT. CFTs with the $N = 2$ supersymmetry have been actively studied mainly due to possible applications to string theories. In particular, the $N = 2$ SLFT appears in the context of the black-hole solutions of a string theory [7]. With or without the conformal symmetry, two-dimensional models with $N = 2$ supersymmetry show an interesting feature, namely, the nonrenormalization. The parameters in the supersymmetric action do not change in all orders of perturbative calculations. This means that the $N = 2$ SLFT maintains the conformal symmetry without renormalization of the background charge and loses the self-duality.

Our main proposal in this paper is that the $N = 2$ SLFT still shows an interesting duality behaviour. Under the dual transformation $b \to 1/b$, the theory maps to a 'dual' action which is another $N = 2$ super CFT. The $N = 2$ SLFT with a strong coupling can be described by the dual action perturbatively. We compute the reflection amplitudes (the two-point functions) of the theory using functional relations derived from the actions. This procedure provides an exact relation between two parameters $\mu, \tilde{\mu}$, which fixes the relations between the two actions completely. To check a self-consistency of our proposal, we compare the reflection amplitudes derived from the conjecture action with some independent result derived in a totally different context.

2 $N = 2$ Super-Liouville theory and its Dual action

The action of the $N = 2$ SLFT at the flat background is given by

$$\mathcal{A}_{\mathrm{I}}(b) = \int d^2z \left[\int d^4\theta S S^\dagger + \mu \int d^2\theta e^{bS} + c.c. \right] \tag{1}$$

where S is a scalar superfield satisfying

$$D_- S = \overline{D}_- S = 0, \qquad D_+ S^\dagger = \overline{D}_+ S^\dagger = 0. \tag{2}$$

The Lagrangian of the $N=2$ SLFT can be expressed in terms of the component fields as follows:

$$\mathcal{L} = \frac{1}{4\pi}\left[2\varphi\partial\overline{\partial}\varphi^\dagger + 2\varphi^\dagger\partial\overline{\partial}\varphi + \psi^\dagger\overline{\partial}\psi + \psi\overline{\partial}\psi^\dagger + \overline{\psi}^\dagger\partial\overline{\psi} + \overline{\psi}\partial\overline{\psi}^\dagger\right] \tag{3}$$

$$- \frac{5}{4}\pi\mu^2 b^2 e^{b\varphi+b\varphi^\dagger} + \frac{\mu b^2}{2}\psi^\dagger\overline{\psi}^\dagger e^{b\varphi} + \frac{\mu b^2}{2}\psi\overline{\psi}e^{b\varphi^\dagger}. \tag{4}$$

As in the LFT and the $N=1$ SLFTs, one should introduce a background charge $1/b$ so that the second term in Eq.(1) becomes the screening operator of the conformal field theory (CFT). However, a fundamental difference arises where the background charge is unrenormalized due to the $N=2$ supersymmetry. For the LFT and the $N=1$ SLFTs, the background charge is renormalized to $Q=1/b+b$ and the theories are invariant under the dual transformation $b\to 1/b$. This self-duality plays an essential role to determine various exact correlation functions of those Liouville theories. Unrenormalized, the $N=2$ SLFT is not self-dual.

This theory is a CFT with a central charge

$$c = 3 + 6/b^2. \tag{5}$$

The primary operators of the $N=2$ SLFT are classified into the Neveu-Schwarz (NS) and the Ramond (R) sectors and can be written in terms of the component fields as follows [8]:

$$N_{\alpha\overline{\alpha}} = e^{\alpha\varphi^\dagger+\overline{\alpha}\varphi}, \qquad R^{\pm}_{\alpha\overline{\alpha}} = \sigma^{\pm}e^{\alpha\varphi^\dagger+\overline{\alpha}\varphi}, \tag{6}$$

where $\sigma^\pm$ is the spin operators. The conformal dimensions of these fields are given by

$$\Delta^N_{\alpha\overline{\alpha}} = -\alpha\overline{\alpha} + \frac{1}{2b}(\alpha+\overline{\alpha}), \qquad \Delta^R_{\alpha\overline{\alpha}} = \Delta^N_{\alpha\overline{\alpha}} + \frac{1}{8}. \tag{7}$$

The $U(1)$ charges are given by

$$Q^N_{\alpha\overline{\alpha}} = -\frac{1}{2b}(\alpha-\overline{\alpha}), \qquad Q^{R\pm}_{\alpha\overline{\alpha}} = Q^N_{\alpha\overline{\alpha}} \pm \frac{1}{4}. \tag{8}$$

From these expressions, one can notice that

$$\alpha \to 1/b - \overline{\alpha}, \qquad \overline{\alpha} \to 1/b - \alpha \tag{9}$$

do not change the same conformal dimension and $U(1)$ charge. From the CFT point of view, this means that $N_{1/b-\overline{\alpha},1/b-\alpha}$ should be identified with $N_{\alpha\overline{\alpha}}$ and similarly for the (R) operators upto normalization factors. The reflection amplitudes are determined by these normalization factors.

Without the self-duality, it is possible that there exists a 'dual' action to (1) whose perturbative (weak coupling) behaviours describe the $N=2$ SLFT in the strong coupling region. This action should be another CFT. Our proposal for the dual action is as follows:

$$\mathcal{A}_{\rm II}(b) = \int d^2z \int d^4\theta \left[SS^\dagger + \tilde{\mu}e^{b(S+S^\dagger)}\right] \tag{10}$$

with the background charge b. The $N = 2$ supersymmetry is preserved because $S + S^\dagger$ is a $N = 2$ scalar superfield. One can see that this action is conformally invariant because the interaction term is a screening operator. Our conjecture is that the two actions, $\mathcal{A}_{\rm I}(b)$ and $\mathcal{A}_{\rm II}(1/b)$ are equivalent. To justify this conjecture, we will compute the reflection amplitudes based on these actions and will compare with some independent results.

3 The Reflection Amplitudes

As mentioned above, the reflection amplitudes of the Liouville-type CFT is defined by the linear transformations between different exponential fields, corresponding to the same primary field of chiral algebra. In this paper, we apply the method developed in [9] to derive the reflection amplitudes of the $N = 2$ SLFT. For simplicity, we will restrict ourselves to the case of $\alpha = \overline{\alpha}$ in Eq.(6) where the $U(1)$ charge of the (NS) operators becomes 0. We will refer to this case as the 'neutral' sector. (From now on, we will suppress the second indices $\overline{\alpha}$.) The physical states in this sector are given by

$$\alpha = \frac{1}{2b} + iP \tag{11}$$

where P is a real parameter. This parameter is transformed by $P \to -P$ under the reflection relation (9) and can be thought of as a 'momentum' which is reflected off from a potential wall.

Two-point functions of the same operators can be expressed as

$$\langle N_\alpha(z,\overline{z}) N_\alpha(0,0)\rangle = \frac{D^N(\alpha)}{|z|^{4\Delta^N_\alpha}} \tag{12}$$

$$\langle R^+_\alpha(z,\overline{z}) R^-_\alpha(0,0)\rangle = \frac{D^R(\alpha)}{|z|^{4\Delta^R_\alpha}} \tag{13}$$

where $\Delta^N_\alpha, \Delta^R_\alpha$ are given by Eq.(7). The reflection amplitudes are given by the normalization factors $D^N(\alpha), D^R(\alpha)$ and should satisfy

$$D^N(\alpha)D^N(1/b-\alpha) = 1, \qquad D^R(\alpha)D^R(1/b-\alpha) = 1. \tag{14}$$

To find these amplitudes explicitly, we consider the operator product expansions (OPEs) with degenerate operators.

The (NS) and the (R) degenerate operators in the neutral sector are $N_{\alpha_{nm}}$ and $R^\pm_{\alpha_{nm}}$ with integers n, m and

$$\alpha_{nm} = \frac{1-n}{2b} - \frac{mb}{2}, \qquad n, m \geq 0. \tag{15}$$

The OPE of a (NS) field with a degenerate operator $N_{-b/2}$ is simply given by

$$N_\alpha N_{-b/2} = N_{\alpha-b/2} + C^N_-(\alpha) N_{\alpha+b/2}. \tag{16}$$

Here the structure constant can be obtained from the screening integral as follows:

$$C_-^N(\alpha) = \kappa_1 \gamma(1-\alpha b)\gamma(1/2-\alpha b - b^2/2)\gamma(-1/2+\alpha b)\gamma(\alpha b + b^2/2)\,, \tag{17}$$

where

$$\kappa_1 = \frac{\mu^2 b^4 \pi^2}{2}\gamma(-b^2-1)\gamma\left(1+\frac{b^2}{2}\right)\gamma\left(\frac{b^2}{2}+\frac{3}{2}\right)$$

with $\gamma(x) = \Gamma(x)/\Gamma(1-x)$ as usual.

To use this OPE, we consider a three-point function $\langle N_{\alpha+b/2} N_\alpha N_{-b/2}\rangle$ and take the OPE by $N_{-b/2}$ either on $N_{\alpha+b/2}$ or on N_α using Eq.(16). This leads to a functional equation

$$C_-^N(\alpha) D^N(\alpha + b/2) = D^N(\alpha). \tag{18}$$

This functional equation can determine the (NS) reflection amplitude in the following form:

$$D^N(\alpha) = (\frac{\kappa_1}{b^4})^{-2\alpha/b}\gamma(2\alpha/b - 1/b^2)\frac{\gamma(b\alpha+1/2)}{\gamma(b\alpha)} f(\alpha) \tag{19}$$

with an aribitrary function $f(\alpha)$ satisfying $f(\alpha) = f(\alpha+b)$. To fix this unknown function, we need an additional functional equation. It is natural that this relation is provided by the dual action $\mathcal{A}_{\rm II}(1/b)$.

For this purpose, we consider OPEs with another degenerate operator, namely,

$$N_\alpha R^+_{-1/2b} = R^+_{\alpha-1/2b} + \tilde{C}^N_-(\alpha) R^+_{\alpha+1/2b} \tag{20}$$
$$R^-_\alpha R^+_{-1/2b} = N_{\alpha-1/2b} + \tilde{C}^R_-(\alpha) N_{\alpha+1/2b}. \tag{21}$$

The structure constants can be computed by the screening integrals using the dual action $\mathcal{A}_{\rm II}(1/b)$, which is equivalent to $\mathcal{A}_{\rm I}(b)$. The results are

$$\tilde{C}^N_-(\alpha) = \kappa_2(b)\frac{\gamma(2\alpha/b - 1/b^2)}{\gamma(2\alpha/b)}, \tag{22}$$
$$\tilde{C}^R_-(\alpha) = \kappa_2(b)\frac{\gamma(2\alpha/b - 1/b^2+1)}{\gamma(2\alpha/b+1)} \tag{23}$$

where

$$\kappa_2(b) = \tilde{\mu}\pi\gamma\left(\frac{1}{b^2}+1\right)\,. \tag{24}$$

These results are consistent with the $N=2$ superminimal CFT results [8].

Now we consider three-point functions, $\langle R^-_{\alpha+1/2b} N_\alpha R^+_{-1/2b}\rangle$ and $\langle N_{\alpha+1/2b} R^-_\alpha R^+_{-1/2b}\rangle$. Taking OPE with $R^+_{-1/2b}$ on one of two other operators in the correlation functions and using the OPE relations (20) and (21), we obtain an independent set of functional relations as follows:

$$\tilde{C}_{-}^{N}(\alpha)D^{R}(\alpha+1/2b) = D^{N}(\alpha), \tag{25}$$
$$\tilde{C}_{-}^{R}(\alpha)D^{N}(\alpha+1/2b) = D^{R}(\alpha). \tag{26}$$

Solving for the $D^N(\alpha)$, we find a most general solution of Eqs.(25) is

$$D^{N}(\alpha)=\kappa_2^{-2\alpha b}\frac{\Gamma^2(\alpha b+1/2)}{\Gamma^2(\alpha b)}\gamma(2\alpha/b-1/b^2)g(\alpha) \tag{27}$$

where $g(\alpha)$ is another arbitrary function satisfying $g(\alpha)=g(\alpha+1/b)$. Combining Eqs.(19) and (27), and requiring the normalization $D^N(\alpha=\frac{1}{2b})=1$, we can determine the (NS) reflection amplitude completely as follows:

$$D^{N}(\alpha)=-\frac{2}{b^2}\,\kappa_2^{-2\alpha b+1}\,\gamma\left(\frac{2\alpha}{b}-\frac{1}{b^2}\right)\gamma\left(\alpha b+\frac{1}{2}\right)\gamma(1-\alpha b)\,, \tag{28}$$

where two parameters in the actions, μ and $\tilde{\mu}$, are related by

$$\left(\frac{\kappa_1}{b^4}\right)^{1/b}=\kappa_2^{b}. \tag{29}$$

The (R) reflection amplitude can be obtained by Eq.(25):

$$D^{R}(\alpha)=-\frac{b^2}{2}\kappa_2^{-2\alpha b+1}\,\gamma\left(\frac{2\alpha}{b}-\frac{1}{b^2}+1\right)\gamma\left(-\alpha b+\frac{1}{2}\right)\gamma(\alpha b). \tag{30}$$

We can rewrite these amplitudes using the momentum P defined in Eq.(11):

$$D^{N}(P) = \kappa_2^{-2iPb}\frac{\Gamma\left(1+\frac{2iP}{b}\right)}{\Gamma\left(1-\frac{2iP}{b}\right)}\frac{\Gamma\left(1+iPb\right)}{\Gamma\left(1-iPb\right)}\frac{\Gamma\left(\frac{1}{2}-iPb\right)}{\Gamma\left(\frac{1}{2}+iPb\right)}, \tag{31}$$
$$D^{R}(P) = \kappa_2^{-2iPb}\frac{\Gamma\left(1+\frac{2iP}{b}\right)}{\Gamma\left(1-\frac{2iP}{b}\right)}\frac{\Gamma\left(1-iPb\right)}{\Gamma\left(1+iPb\right)}\frac{\Gamma\left(\frac{1}{2}+iPb\right)}{\Gamma\left(\frac{1}{2}-iPb\right)}. \tag{32}$$

4 Consistency Check

To justify the reflection amplitudes derived in the previous section based on the conjectured action $\mathcal{A}_{\rm II}$, we provide several consistency checks.

It has been noticed that an integrable model with two parameters proposed in [10] can have $N=2$ supersymmetry if one of the parameters take a special value [11]. This means that one can compute the reflection amplitudes of the $N=2$ SLFT independently as a special case of those in [11]. Indeed, we have confirmed that the two results agree exactly.

Furthermore, one can check the reflection amplitude for specicific values of α directly from the action. When $\alpha=\frac{1}{2b}-\frac{b}{2}$, the Coulomb integral using the action $\mathcal{A}_{\rm I}(b)$ gives

$$\begin{aligned}\langle N_\alpha(0)N_\alpha(1)\rangle &= (\frac{\mu b^2}{2})^2\int d^2z_1d^2z_2\langle e^{\alpha(\varphi+\varphi^\dagger)}(0)\,e^{\alpha(\varphi+\varphi^\dagger)}(1)\,\psi^\dagger\overline{\psi}^\dagger e^{b\varphi(z_1,\bar{z}_1)}\,\psi\overline{\psi}\,e^{b\varphi^\dagger(z_2,\bar{z}_2)}\rangle \\ &= \frac{\mu^2b^3\pi^2}{(\alpha-\frac{1-b^2}{2b})}\gamma(\frac{1+b^2}{2})\frac{\gamma(-1-b^2)}{\gamma(-1/2-b^2/2)}\,. \end{aligned} \tag{33}$$

This result agrees with Eq.(28) for $\alpha\to\frac{1}{2b}-\frac{b}{2}$. Similarly, when $\alpha\to 0$, one can compute the two-point function directly from the action $\mathcal{A}_{\rm II}(1/b)$ and can get

$$\begin{aligned}\langle N_\alpha(0)N_\alpha(1)\rangle &= -\frac{\tilde{\mu}}{b^2}\int d^2z\langle e^{\alpha(\varphi+\varphi^\dagger)}(0)\,e^{\alpha(\varphi+\varphi^\dagger)}(1)\,e^{(\varphi+\varphi^\dagger)/b}\partial(\varphi+\varphi^\dagger)\bar{\partial}(\varphi+\varphi^\dagger)(z,\bar{z})\rangle \\ &= -\frac{\tilde{\mu}}{b^2}\pi\alpha b=0 \qquad \text{as} \quad \alpha\to 0\,. \end{aligned} \tag{34}$$

Here, the insertion we have considered is the only term in the action $\mathcal{A}_{\rm II}(1/b)$ which can give nonvanishing contribution. Again, this result agrees with Eq.(28) for $\alpha=0$.

In the semiclassical limit $b\to 0$, the reflection amplitudes can be interpreted as the quantum mechanical reflection amplitudes of the wave function of the zero-modes from the exponential potential wall arising from the action (1). One can easily find that the reflection amplitude corresponding to the (NS) operator is given by

$$R^N(P)\sim\frac{\Gamma(1+2iP/b)}{\Gamma(1-2iP/b)}, \tag{35}$$

which is consistent with Eq.(32) in the limit $b\to 0$.

In summary, we have conjectured a strong coupling effective action which is dual to the $N=2$ SLFT. Based on this conjecture, we have computed the reflection amplitudes (the two-point functions) of the (NS) and the (R) primary fields exactly. We have fixed the relation between the two parameters μ and $\tilde{\mu}$. Then, we have checked the validity of these amplitudes by comparing with an independent result along with some other consistency checks. It would be nice to provide more stringent check. One possibility is to consider the $N=2$ supersymmetric sinh-Gordon (or sine-Gordon) model as a perturbed integrable model of the $N=2$ SLFT. Being integrable, one can compare the finite-size correction of the central charge either by the thermodynamic Bethe ansatz or by the quantization conditions based on the conjecture reflection amplitudes [12].

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